Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 6, pp. 1061–1068, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)00135-9

INTERACTION OF COUPLED MODES ACCOMPANYING NON-LINEAR FLEXURAL VIBRATIONS OF A CIRCULAR RING†

A. I. MANEVICH

Dnepropetrovsk

(*Received* 19 *April* 1993)

An analytic investigation of the problem of the non-linear free flexural vibrations of a circular ring taking account of the coupled modes is presented. A system of amplitude–frequency modulation equations is obtained by the method of many scales. The integral of this system enables one to construct an "amplitude–phase pattern" which characterizes the possible dynamic modes in the case of arbitrary initial conditions. It is shown that an energy threshold exists and, when this is exceeded, the appearance of a travelling wave and pronounced amplitude–frequency modulation are possible. The threshold value of the energy depends on the "detuning" of the frequencies of the coupled modes due to the presence of initial imperfections. A general solution of the problem is obtained in elliptic Jacobi functions.

In the case of flexural vibrations of circular rings, shells of revolution and disks, the directly excited fundamental modes may be accompanied by the appearance of "coupled" modes (which are geometrically similar but shifted in a circumferential direction by an angle $\varphi = \pi/2n$) where n is the number of circumferential waves). This leads to the appearance of travelling waves [1–3]. The mathematical model of vibrations constructed in [4] has enabled the non-linear nature of these phenomena to be established. It has been shown that the free vibrations of rings and shells are amplitude–frequency modulated vibrations with periodic energy exchange between coupled modes [2]. However, there has been no complete analytic investigation of the problem up to the present time. The interesting analytic model of the non-linear interaction of vibrations in a ring [5] describes a somewhat different phenomenon, that is energy exchange between radial extension–compression modes and flexural modes (the interaction of three modes). Radii extension–compression modes were not excited in the experiments in [1, 2], and an explanation of the results in these papers needs to be sought within the framework of an analytic model which describes the interaction of coupled modes.

1. INITIAL EQUATIONS. SYSTEM OF AMPLITUDE–FREQUENCY MODULATION EQUATIONS AND ITS INTEGRAL

Consider the free flexural vibrations of a ring of radius R in its plane (with the axis of symmetry of the cross-section lying in this plane). We take the radial flexure in the form (a positive flexure is in the direction of the outward normal)

$$w = f_1 \cos n\varphi + f_2 \sin n\varphi + f_0 \tag{1.1}$$

where $\varphi = y/R$ is an angular coordinate. The component f_0 takes account of the radial displacement of the axial line of the ring as a consequence of flexure and is determined from the condition that the mean membrane stress (when there is no axially symmetric extension–compression strain) is equal to zero. The axial strain ε and the change of curvature χ are defined by the expressions (see [6], for example)

$$\varepsilon = (w + \upsilon_{,\varphi}) / R + (w_{,\varphi} - \upsilon)^2 / (2R^2), \quad \chi = (w_{,\varphi} - \upsilon)_{,\varphi} / R^2$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 6, pp. 119–125, 1994.

(υ is the tangential displacement). It follows from the condition $\int_0^{2\pi} \varepsilon R\, d\varphi = 0$, which is satisfied exactly when there is no axially symmetric strain, and when (1.1) and the periodicity of υ with respect to φ are taken into account that f_0 has a quadratic order

$$f_0 = -\frac{1}{4\pi R}\int_0^{2\pi} (w,_{\varphi} - \upsilon)^2 d\varphi$$

The tangential displacement υ is determined from the condition $w + \upsilon_{,\varphi} = 0$ which follows from the smallness of ε and $(w_{,\varphi} - \upsilon)^2/(2R^2)$ compared with the linear terms in the expression for ε (the approximation in the determination of υ is justified by the smallness of the contribution of υ in the following solution and, when all terms in υ are neglected, the error is of the order of $1/n^2$)

$$\upsilon = (-f_1 \sin n\varphi + f_2 \cos n\varphi)/n$$

We then obtain

$$f_0 = -\frac{(n^2-1)^2}{4Rn^2}(f_1^2 + f_2^2) \tag{1.2}$$

(the factor n^2 is given in [1] and other papers instead of $n^2 - 1$).

From Lagrange's equations, on defining the potential and kinetic energy by

$$V = \frac{E\dot{I}}{2}\int_0^{2\pi} \chi^2 R d\varphi = \frac{\pi E\dot{I}(n^2-1)^2}{2R^3}(f_1^2 + f_2^2)$$

$$T = \frac{\rho A}{2}\int_0^{2\pi} (\dot{w}^2 + \dot{\upsilon}^2) R d\varphi = \frac{\pi\rho A R}{2}\left[\dot{f}_0^2 + \frac{n^2+1}{n^2}(\dot{f}_1^2 + \dot{f}_2^2)\right] \tag{1.3}$$

(ρ, A and $\dot{I}$ are the density, the area and moment of inertia of the cross-section; differentiation with respect to t is denoted by a dot), we obtain a system of equations in f_1 and f_2 containing cubic nonlinearities

$$\ddot{f}_k + \omega_k^2 f_k + 2\kappa f_k(f_1\ddot{f}_1 + \dot{f}_1^2 + f_2\ddot{f}_2 + \dot{f}_2^2) = 0, \quad k = 1,2 \tag{1.4}$$

$$\left(\omega_1^2 = \omega^2 = \frac{E\dot{I}(n^2-1)^2 n^2}{\rho A R^4(n^2+1)}, \quad 2\kappa = \frac{(n^2-1)^4}{2n^4R^2}\right)$$

For generality in the analysis, the value of ω_2 is assumed not to be the same as the value of ω (unlike in [4]) although it is close in magnitude to ω. When there are initial imperfections, the eigenfrequency spectrum of the coupled modes decomposes [7], and it is therefore assumed that

$$\omega_2^2 = \omega^2 + \varepsilon^2\sigma \tag{1.5}$$

where ε is a small parameter and σ is the frequency "detuning" parameter. We note that imperfections also lead to the appearance of a linear relation between coupled modes [2]. However, under conventional experimental conditions (where the position of the nodes is not fixed) fundamental vibrational modes are excited for which there is no linear relation by virtue of their orthogonality (that is, the arrangement of the nodes and anti-nodes is determined by the modes with the extremum values of the characteristic frequencies).

The system of equations (1.4) is solved by the method of many scales [8]. On introducing "fast" and "slow" times $T_0 = t$, $T_s = \varepsilon^s T_0$ ($s = 1, 2, \ldots$) and seeking the solution of this system in the form of an expansion

$$f_k = \varepsilon f_{k1} + \varepsilon^3 f_{k3} + \ldots, \quad k = 1,2 \tag{1.6}$$

we obtain, after applying the standard procedure, the equations for two approximations

$$D_0^2 f_{k1} + \omega^2 f_{k1} = 0, \quad k = 1,2 \tag{1.7}$$

$$D_0^2 f_{k3} + \omega^2 f_{k3} + (2D_0 D_2 + D_1^2) f_{k1} +$$
$$+ 2\kappa f_{k1}[f_{11} D_0^2 f_{11} + (D_0 f_{11})^2 + f_{21} D_0^2 f_{21} + (D_0 f_{21})^2] + \sigma \delta_{k2} f_{k1} = 0 \tag{1.8}$$

where $D_0 = \partial/\partial T_0$, $D_s = \partial/\partial T$ and δ_{ks} is the Kronecker delta.

We will write the solution of system (1.7) in the form (cc is a complex conjugate quantity)

$$f_{k1} = A_k(T_2)e^{i\omega T_0} + \text{cc}, k = 1, 2. \tag{1.9}$$

Substituting (1.9) into (1.8), we obtain from the condition that there are no secular terms (differentiation with respect to the slow time T_2 is denoted by a prime)

$$-2i\omega A_k' + 4\kappa\omega^2 (A_1^2 + A_2^2)\bar{A}_k - \delta_{k2}\sigma A_2 = 0, \quad k = 1,2 \tag{1.10}$$

On changing to the exponential form for the complex amplitudes

$$A_k = \frac{1}{2} a_k e^{i\theta_k} \tag{1.11}$$

and, on separating the real and imaginary parts in (1.10), we obtain a system of equations which determines the modulation of the amplitudes and phases of the coupled modes

$$(a_k^2)' + (-1)^k \kappa\omega a_1^2 a_2^2 \sin 2\gamma = 0, \quad k = 1,2 \tag{1.12}$$
$$2\theta_1' + \kappa\omega(a_1^2 + a_2^2 \cos 2\gamma) = 0$$
$$2\theta_2' + \kappa\omega(a_1^2 \cos 2\gamma + a_2^2) - \sigma/\omega = 0 \quad (\gamma = \theta_2 - \theta_1)$$

From the first two equations of (1.12), we obtain the energy integral

$$a_1^2 + a_2^2 = e \tag{1.13}$$

(the arbitrary constant e is proportional to the energy of the system).

We now introduce the variable ξ by the relation $\xi = a_1^2/e (0 \leqslant \xi \leqslant 1)$. Then, $a_2^2 = e(1-\xi)$ and, from (1.12), we obtain the following system of equations in ξ and γ (the second equation is obtained by subtracting the penultimate equation of (1.12) from the last equation of (1.12))

$$\xi' = \kappa\omega e \xi(1-\xi)\sin 2\gamma$$
$$2\gamma' = -\kappa\omega e(1-2\xi)(1-\cos 2\gamma) + \sigma/\omega \tag{1.14}$$

On dividing the first equation of (1.14) by the second, we obtain an equation in complete differentials which has the integral

$$\xi(1-\xi)(1-\cos 2\gamma) - \sigma^* \xi = C, \quad \sigma^* = \sigma/(e\omega^2\kappa) \tag{1.15}$$

On the other hand, on eliminating γ from the first equation of (1.14) and the integral (1.15), we obtain an equation which determines the dependence of ξ on the slow time T_2

$$(e\omega\kappa)^{-2}(\xi')^2 = (\sigma^*\xi + C)[2\xi(1-\xi) - (\sigma^*\xi + C)] \tag{1.16}$$

2. THE AMPLITUDE–PHASE PATTERN OF THE SYSTEM. STEADY STATES. STABILITY. THE DEPENDENCE OF THE MODE OF VIBRATION ON THE ENERGY AND THE DETUNING OF THE FREQUENCIES OF COUPLED MODES

We will first consider what conclusions regarding the behaviour of a ring can be drawn on the basis of the integral (1.15). By virtue of the periodicity with respect to γ, it is possible to confine oneself in the analysis to a treatment of the rectangle $0 \leqslant \xi \leqslant 1, 0 \leqslant \gamma \leqslant \pi$ in the (ξ, γ)-plane. The integral curves (1.15) in this rectangle constitute the "amplitude–phase pattern" (APP) of the system, which provides a clear representation of the modes of vibration in the case of arbitrary initial conditions and their stabilities. The topography of an APP is determined by the number of stationary points (1.14) (points of the extrema C (1.15)) which correspond to stationary, that is, single-frequency, vibrational conditions. The stationary points are defined by the system of equations

$$\xi(1-\xi)\sin 2\gamma = 0, \quad (1-2\xi)(1-\cos 2\gamma)-\sigma^* = 0 \tag{2.1}$$

which has the following solutions

$$\begin{gathered}(\mathrm{a})\ \xi = 0, \quad \gamma = \gamma_s^-; (\mathrm{b})\ \xi = 1, \quad \gamma = \gamma_s^+ \\ \gamma_s^\pm = (\tfrac{1}{2})(-1)^s \arccos(1 \pm \sigma^*) + s\pi, \quad s = 0,1;\ (\mathrm{c})\ \xi = \tfrac{1}{2} - \sigma^*/4, \quad \gamma = \pi/2\end{gathered} \tag{2.2}$$

These stationary points exist when the following conditions are respectively satisfied

$$(\mathrm{a})\ \ 0 \leqslant \sigma^* \leqslant 2,\ (\mathrm{b}) -2 \leqslant \sigma^* \leqslant 0,\ (\mathrm{c})\ |\sigma^*| \leqslant 2 \tag{2.3}$$

Elementary analysis shows that the points $(0, \gamma_s^-)$ and $(1, \gamma_s^+)$ $(s = 1, 2)$, to which vibrations of only one each of the coupled modes correspond, are unstable and that the last point is unstable. Without loss of generality, it may be assumed that $\sigma^* \geqslant 0$ (otherwise, it suffices to change the numbering of the coupled modes). Then, depending on the magnitude of σ^*, the two types of APP shown in Fig. 1 are possible: (a) $\sigma^* \leqslant 2$, (b) $\sigma^* > 2$ (only half of the APP is shown by virtue of its symmetry about the line $\gamma = \pi/2$). The separatrix (the dashed line), which joins the unstable stationary points and encompasses the stable point, separates two domains; one with a difference in the phases of the coupled modes which oscillates with respect to $\gamma = \pi/2$ and the other with a monotonically varying phase difference (the equation of the separatrix is obtained from (1.15) with $C = 0$ and $C = -\sigma^*$, respectively).

In the case of an ideal ring ($\sigma^* = 0$), there is a stable stationary point at $\xi = \frac{1}{2}, \gamma = \pi/2$ and unstable "stationary lines" at $\gamma = 0$ and $\gamma = \pi$. The APP for this case is shown in Fig. 2 (the constant C takes values from 0 to $\frac{1}{2}$). The in-phase ($\gamma = 0$) or out-of-phase ($\gamma = \pi$) vibrations of the coupled modes are stationary for any ratio of the amplitudes (for any ξ), but these vibrations are unstable. Vibrations with the same amplitudes of the coupled modes and with a phase difference $\gamma = \pi/2$ constitute the sole stable steady state. On substituting $f_1 = \varepsilon a \cos \omega t, f_2 = \varepsilon a \sin \omega t$ into (1.1), we obtain

$$w = \varepsilon\left[a\cos(\omega t - n\varphi) - \frac{a^2(n^2-1)^2}{2Rn^2}\right] \tag{2.4}$$

which corresponds to a "fast" travelling wave (with the frequency of the characteristic vibrations of the ring ω). All the remaining modes of vibration are amplitude–frequency modulated vibrations (they may be considered to be the result of the superposition of slow modulation waves on a fast travelling wave, and the ratio of the amplitudes of the fast and slow components depends on the closeness of the curve to the central point).

The following physical explanation can be given for the stability of the travelling wave ($\gamma = \pi/2$) and the instability of the in-phase or out-of-phase vibrations. Let us consider the radial component of the vibrations in expression (1.1) which is defined by (1.2) for a certain phase difference γ. Putting $f_1 = \varepsilon a_1 \cos \omega t, f_2 = \varepsilon a_2 \cos(\omega t + \gamma)$, we obtain

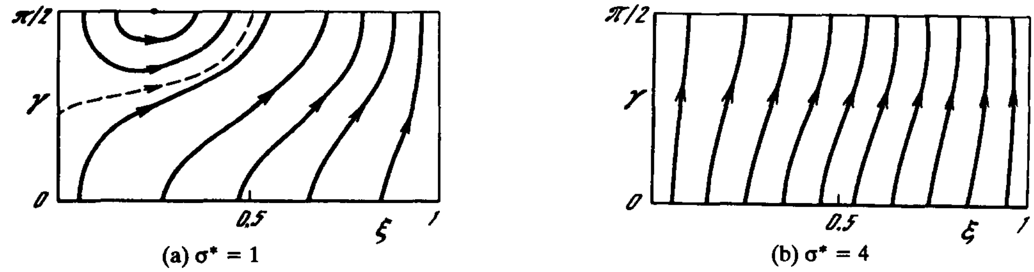

Fig. 1.

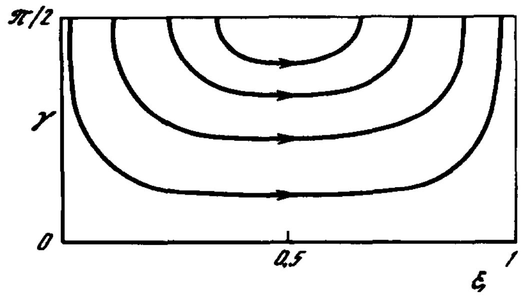

Fig. 2.

$$f_1^2 + f_2^2 = \varepsilon^2[a_1^2 + a_2^2 + a_1^2 \cos 2\omega t + a_2^2 \cos 2(\omega t + \gamma)]/2 \qquad (2.5)$$

The amplitude of the radial vibrations is a maximum when $\gamma = 0$ and $\gamma = \pi$ and a minimum (it is proportional to $|a_1^2 - a_2^2|$) when $\gamma = \pi/2$. The travelling-wave mode for an ideal ring ($a_1 = a_2$) is the unique mode for which there are no radial vibrations of the axial line.

Returning to the general case of an ideal ring, we note that, in the case of the unique stable stationary point (when $\gamma = \pi/2$) $a_1 \neq a_2$, and the coupled mode with the higher characteristic frequency has the larger amplitude. The corresponding mode of vibration can be considered as the superposition of a fast wave of amplitude a_1 and a standing wave of amplitude $a_2 - a_1$ (when $a_1 < a_2$) where the fast component tends to zero as σ^* increases. The "domain of attraction" of the stationary point, bounded by the separatrix, simultaneously decreases and disappears when $\sigma^* = 2$ (three stationary points merge).

When account is taken of the relationship for σ^* in (1.15), the condition $\sigma^* \leqslant 2$ can be written in the form

$$e \geqslant \sigma/(2\kappa\omega^2) \qquad (2.6)$$

Let us write this condition in terms of quantities which are independent of the arbitrarily chosen parameter ε. Taking (1.5), (1.6), (1.9), (1.11) and (1.13) into account, we have, putting $\theta_k(0) = 0$, $k = 1, 2$

$$E_0 = f_1^2(0) + f_2^2(0) \geqslant E^* = \frac{\omega_2^2 - \omega^2}{2\kappa\omega^2} = \frac{\Delta\omega}{\kappa\omega} \quad (\Delta\omega = \omega_2 - \omega) \qquad (2.7)$$

This expression shows that the form of the possible modes of vibration is determined by the energy of the vibrations. For a given value of $\Delta\omega$, a threshold value E^* exists, and, when this value is exceeded, an APP of type *b* (Fig. 1b) changes into a picture of type *a* (Fig. 1a), a pronounced interaction of the coupled modes appears and a travelling wave mode becomes possible (the threshold value E^* tends to zero as $\Delta\omega$ decreases).

This result explains the experimental observations in the case of cylindrical shells [2]: a strong modulation (beats) is observed when free vibrations of large amplitude (of the order of 5–10 thicknesses) are excited, but the beats disappear when the amplitudes decrease (due to damping). When $f_2(0) = 0$ we obtain from (2.7), taking account of (1.4) (h is the thickness of the ring or shell)

$$\frac{f_1(0)}{h} \geqslant \frac{2n^2}{(n^2-1)^2}\frac{R}{h}\sqrt{\frac{\Delta\omega}{\omega}} \tag{2.8}$$

For example, for one of the shells in [2] with $h/R = 3.125 \times 10^{-3}$, $n = 4$, $\omega = 2\pi \times 36.9$, $\omega_2 = 2\pi \times 37.8$, we obtain $f_1(0)/h \geqslant 7$, which is in good agreement with experimental observations.

3. DEPENDENCE OF THE AMPLITUDES AND FREQUENCIES ON TIME. THE GENERAL SOLUTION

The dependence of the amplitudes of coupled modes on the slow time is defined by Eq. (1.16). By virtue of the positiveness of the right-hand side of (1.16), segments of the lines $y = \sigma^*\xi + C$ lying within the domain bounded by the parabola $y = 2\xi(1-\xi)$ (Fig. 3) correspond to solutions. In the case of an ideal ring ($\sigma^* = 0$), the inclined lines become lines parallel to the abscissa.

We will first consider the case when $\sigma^* = 0$. Equation (1.16) reduces to

$$d\xi / dT_2 = \pm\sqrt{2C}e\kappa\omega\sqrt{(\xi-\xi_1)(\xi_2-\xi)} \tag{3.1}$$

where ξ_1, ξ_2 are the roots of the trinomial in the square brackets of (1.16): $\xi_2 = 1 - \xi_1$, $C = 2\xi_1(1-\xi_1)$ and, subject to the initial condition $\xi(0) = \xi_0$, has the solution

$$\xi = \tfrac{1}{2} + (\tfrac{1}{2} - \xi_1)\sin(eDT_2 + \alpha_0), \quad D = \pm 2\kappa\omega\Lambda \tag{3.2}$$

$$\Lambda = \sqrt{\xi_1(1-\xi_1)}, \quad \alpha_0 = \arcsin\frac{2\xi_0 - \xi_1 - \xi_2}{\xi_2 - \xi_1}$$

Substituting the solution obtained into the second equation of (1.14) and taking account of the integral (1.15), we determine the phase difference

$$\gamma = \gamma_0 + \frac{\sigma T_2}{2\omega} \mp \left[\operatorname{arctg}\frac{(1-2\xi_1)\cos(eDT_2+\alpha_0)}{2\Lambda} - \alpha_1\right] \tag{3.3}$$

$$\alpha_1 = \operatorname{arctg}\frac{(1-2\xi_1)\cos\alpha_0}{2\Lambda}, \quad \gamma_0 = \gamma(0)$$

and, we then find θ_1 and θ_2 from the last two equations of (1.12).

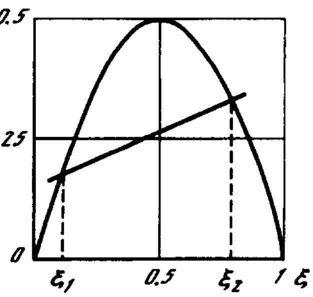

Fig. 3.

For example, we obtain

$$\theta_1 = \theta_{10} - \frac{e\kappa\omega T_2}{2} \pm \left\{ \operatorname{arctg} \frac{\operatorname{tg}[(eDT_2 + \alpha_0)/2] - (1 - 2\xi_1)}{2\Lambda} - \alpha_2 \right\} \tag{3.4}$$

$$\alpha_2 = \operatorname{arctg} \frac{\operatorname{tg}(\alpha_0/2) - (1 - 2\xi_1)}{2\Lambda}, \quad \theta_{10} = \theta_1(0)$$

On calculating $a_1 = \sqrt{(e\xi)}$, $a_2 = \sqrt{(e(1-\xi))}$ and changing to real time t and the energy parameter $E_0 = \varepsilon^2 e$ which is independent of ε, we obtain, using (1.4)

$$f_1(t) = \{E_0[\tfrac{1}{2} + (\tfrac{1}{2} - \xi_1)\sin(DE_0 t + \alpha)]\}^{1/2} \cos(\omega t + \theta_1) \tag{3.5}$$

where θ_1 is defined by (3.4) when account is taken of the equality $eT_2 = E_0 t$. A similar expression can also be written in the case of $f_2(t)$.

It can be seen from (3.5) and the expression for κ (1.4) that the modulation period is equal to

$$T^0 = \frac{T^0_{\min}}{2\sqrt{\xi_1(1-\xi_1)}}, \quad T^0_{\min} = \frac{8\pi n^4}{(n^2-1)^4 E_0\omega} \tag{3.6}$$

The period depends on energy of the vibrations and the initial ratio of the amplitudes. In the case of integral curves which approximate to the lines $\gamma = 0$ and $\gamma = \pi$, T^0 tends to infinity while, when approaching the stationary point ($\xi_1 \to \tfrac{1}{2}$, $\gamma \to \pi/2$) it tends to the minimum value $T^0_{\min}$.

We will now consider the general case when $\sigma^* \neq 0$. As previously, we will denote by ξ_1 and ξ_2 the roots of the polynomial on the right-hand side of (1.16) lying in the interval (0, 1) (there can be only two of them). The third root ξ_3 when $\sigma^* < 0$ lies to the right of this interval and, when $\sigma^* > 0$, to the left so that, when $0 \leqslant \xi \leqslant 1$, we have $\sigma^*(\xi - \xi_3) > 0$. Equation (1.16) then reduces to the following (the initial value of ξ is assumed to be equal to the maximum value ξ_2, for simplicity)

$$\int_{\xi_2}^{\xi} [\sigma^*(\xi - \xi_1)(\xi_2 - \xi)(\xi - \xi_3)]^{-1/2} d\xi = \pm\sqrt{2} e\omega\kappa T_2$$

Assuming that $\sigma^* > 0$ and, correspondingly, that $\xi_3 < 0$, by making the substitution

$$\xi = \xi_2 - (\xi_2 - \xi_1)\sin^2\psi$$

we can reduce the integral to an elliptic integral of the first kind and write the solution in terms of elliptic Jacobi functions

$$\xi = \xi_2 - (\xi_2 - \xi_1)\operatorname{sn}^2(z, \eta), \quad z = \mp\left[\frac{e\sigma\kappa(\xi_2 - \xi_3)}{2}\right]^{1/2} T_2, \quad \eta = \left[\frac{\xi_2 - \xi_1}{\xi_2 - \xi_3}\right]^{1/2} \tag{3.7}$$

For the amplitudes of the vibrations, we obtain

$$a_1 = \{e[\xi_2 - (\xi_2 - \xi_1)\operatorname{sn}^2(z, \eta)]\}^{1/2}, \quad a_2 = (e - a_1^2)^{1/2}$$

The modulation period is expressed in terms of a complete elliptic integral of the first kind

$$T^0 = 2[E_0\kappa\omega\Delta\omega(\xi_2 - \xi_3)]^{-1/2} K(\eta)$$

This period depends on the initial conditions and, as calculations show, is two to three orders of magnitude greater than the period of the characteristic vibrations, which agrees with experimental observations [2].

REFERENCES

1. EVENSEN D. A., Non-linear vibrations of circular cylindrical shells. In *Thin-walled Shell Constructions: Theory, Experiment and Design*, pp. 156–176. Mashionstroyeniye, Moscow, 1980.
2. KUBENKO V. D., KOVAL'CHUK P. S. and KRASNOPOL'SKAYA T. S., *Non-linear Interaction of Modes of Flexural Vibration of Cylindrical Shells*. Naukova Dumka, Kiev, 1984.
3. TOBIAS S. A., Non-linear forced vibrations of circular disks. *Engineering* **186**, 4818, 51–56, 1958.
4. EVENSEN D. A., Non-linear flexural vibrations of thin circular rings. *Trans. ASME, Ser. E. J. Appl. Mech.* **33**, 3, 553–560, 1966.
5. KOVRIGIN D. A. and POTAPOV A. I., The non-linear vibrations of a thin ring. *Prikl. Mekh.* **25**, 3, 76–81, 1989.
6. ALFUTOV N. A., *Principles of the Calculation of the Stability of Elastic Systems.* Mashiostroyeniye, Moscow, 1978.
7. TOBIAS S. A., A theory of imperfection for the vibration of elastic bodies of revolution. *Engineering* **172**, 4470, 409–410, 1951.
8. NAYFEH A. H., *Introduction to Perturbation Methods*. Mir, Moscow, 1984.

Translated by E.L.S.